Crossing the Desert – Additional Explanation

The solution in the book says that the last store pile must be 333.333 miles from Mecca. That's not strictly true – other solutions with store piles closer to Mecca are also possible. But no other solution can yield a better result. Here's why.

Let's begin with an example. Suppose you have 2000 bananas at the point 333.333 miles from Mecca. You leave that point with 1000 bananas, travel 100 miles, drop 800 bananas and return to the starting point. You pick up the remaining 1000 bananas and travel the same 100 miles again, arriving with 900 bananas. You now have 1700 bananas at a point that is 233.333 miles from Mecca. It will take three one-way trips from there to the end, for a total of 700 miles. Therefore, you will need all 1700 bananas to get 1000 bananas to the end. There is no disadvantage to creating the extra store pile, but there is no advantage either.

More generally, suppose the last store pile is at a distance d from Mecca. At least three one-way trips must take place between that point and Mecca, so at least $1000 + 3d$ bananas must arrive at that point sooner or later. Moreover, as explained in the book, it is possible to complete the task in exactly 3 trips only if d is within 333.333 miles of Mecca.

Now suppose the second-to-last store pile is at a distance D from the last store pile. By the same reasoning as before, because there must be at least three trips between the second-to-last stop and the last store pile, at least $1000 + 3d + 3D$ bananas must arrive at the second-to-last store pile sooner or later, and again this minimum can be achieved only if $1000 + 3d + 3D \leq 2000$, which means that the second-to-last store pile must also be within 333.333 miles of Mecca. But this is the same minimum number that would be achieved if the last store pile did not exist. In other words, there is no advantage to be obtained by creating an additional store pile between Mecca and the point 333.33 miles from Mecca. So the last store pile may as well be at a distance of 333.333 miles from Mecca, and you will need to get 2000 bananas there sooner or later.

Similar reasoning shows that there is no advantage to creating an intermediate store pile between any of the critical points identified in the solution in the book, i.e., those where there must be a whole multiple of 1000 bananas for an optimal solution.

Furthermore, no advantage can be obtained by changing the order in which bananas arrive and depart from a storage point. To be optimal, the camel must begin each trip toward Mecca (other than the last one beginning in Medina) with exactly 1000 bananas. The solution in the book achieves this, and other solutions may also do so, but no advantage can be gained.

For example, when moving bananas from the 3000 point to the 2000 point (a distance of 200 miles), after two trips (one round trip, one one-way), there will be

1400 bananas available. At that point, one could make a trip to Mecca with 1000 bananas, leave 333.333, and return 333.333 miles, where 400 bananas would remain. Take 200 back 200 miles (leaving 200) and pick up the last 1000 bananas. Turn around, go 200 miles toward Mecca and pick up the remaining 200 bananas, topping off at 1000. Finish the last 333.33 miles, arriving in Mecca with the last 666.667 bananas. This variation accomplishes the same result as the basic solution, but does not improve upon it.

The reason is clear. The last 333.333 miles must be traversed at least three times sooner or later. The previous 200 miles must be traversed at least 5 times, and so on. So any solution that traverses each interval the minimum number of times, thereby consuming the smallest number of bananas, must be optimal.